Some notes about the density of states for a negative pressure matter

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The main goal of this paper is deriving Density of states $g(\epsilon)$ (degeneracy function) per volume for an equation of state (EOS) $p = -\rho$ (we called it dark energy(DE)). We have concluded that thermodynamic quantities such as pressure and energy density are simple functions of temperature, fugacity, curvature and mass of Bosons. Our work has been expressed the origin of some claims about the negativity of the entropy for the scalar fields models of DE.

I: INTRODUCTION

In recent years some works have been done on the thermal properties of the negative pressure matters, especially on the dark energy. The main assumption of all these models is that the DE is a thermal ensemble at a certain temperature with a specific amount of entropy[1]. It would be usable to take temperature as an intrinsic characteristics of DE, namely a barotropic matter with an EOS as $p = f(\rho)$. This fluid must not violate energy conditions. Also we can take both dark matter (DM) and DE as different types of the same matter field with a phase transition in cosmological distances[2]. There is not any reasonable statistical description in negative pressure matters .In some of recent papers, thermal properties of dark energy have been discussed in terms of the assumption that the dark energy substance is a thermal ensemble at a certain temperature with an associated thermo dynamical entropy[1]. It is usually assumed that this temperature is an intrinsic property of DE rather than the temperature of the heat bath fixed by surrounding matter.
Checking the second law of thermodynamics in the context of the black hole physics is another important problem.Also the generalized second law (GSL) must be satisfied by any physical description of the DE even in the new scenarios of the modified gravity in the non relativistic diffeomorphism broken models such as Hořava proposal[29].If we denote that S_{tot} is the total entropy of a system,then GSL guaranties that $\dot{S}_{tot} \geq 0$ for all times.Recently Mubasher Jamil , et.al have shown that[26] in an interactive model for DE , DM and radiation "*the generalized second law is always and generally valid, independently of the specific interaction form of the fluid's equation-of-state parameters and of the background geometry.*".The history of GSL backs to the Unruh and Wald classical work[28]and Bekenstein works on the black holes[31].There exists a recondite relation between the holographic dark energy model and generalized second law of thermodynamics as in [27] and also in quintom dominated universe [30].Holographic scenario for DE is so popular and was investigated by authors both in the context of the usual FRWL cosmology and also in the

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modified ones and in a braneworld picture of universe[25,32,33,36].There is a delicate relation between Bulk brane interaction , holographic dark energy [34], Gauss Bonnet dark energy models[35,38],Stringy inspired tachyon model[37],phantom like regime [39],Observational constraints on holographic dark energy[40], Holographic Chaplygin gas model[41], Holographic Chaplygin DGP Cosmologies[42]and even Holographic Modified Gravity[43].

Some years ago Kulikov and Pronin [3, 15] constructed a simple formulation of a local quantum statistics of Bosonic field in a curved background. Indeed they explained a simple expression for a grand canonical thermodynamic potential density [4] describing locally all thermo chemical properties of gases with a curvature dependent parts [5].Later The high-temperature expansion of the grand thermodynamic potential of non conformably invariant spin-0 and spin-1/2 gases in an arbitrary static spacetime with their boundary is calculated through the method conforming from Kirsten [16].

In this report, we have worked on the statistics of a negative matter with EOS like DE. At first we have obtained the degeneracy distribution function for massive Bosons in this grand ensemble and it has been shown that the density of states for such an exotic matter is a constant function of energy . We have summed over all energy states once more and obtained a closed form for the grand canonical thermodynamic potential density and then we have inferred the expressions for the density of the entropy and other important functions. We have also shown that these thermodynamic equations are generally related with the fugacity , the Ricci scalar and consequently the temperature and the chemical potential. The chemical potential has been associated with a conserved Boson number. We have stated that from a statistical point of view DE violate energy conditions companionship to the claims stated in the recent papers[6].To shunning from this undesirable result we must take both volume and temperature as two dependent variables as Gong and collaborations have shown in [6].

II: LAGRANGIAN OF THE MODEL AND SOME REMARKS ABOUT THE BACKGROUND METRIC

We begin from Lagrangian of a massive scalar field coupled to a gravity which we assume that is non negative. Consider the action[8]

$$S = \int \sqrt{-g} d^4 x (\frac{1}{2} \phi^{;\mu} \phi_{;\mu} - \frac{1}{2}(m^2 + \xi R)\phi^2) \tag{1}$$

where R is the scalar curvature and we assume that the metric of the space time $g_{\mu\nu}$ is static or slowly time variable such that we can take the Ricci scalar, time independent. Spherical symmetry is not essential in this formalism, because we use the general form for the metric, and only we impose a restriction on the time variation of the metric. The only essential assumption is on the surface tension energy portion in EOS. In the case of bulk matter we ignore the surface effects and all of the thermodynamics quantities are expressed per unit volume. Considering the equilibrium assumption , a static metric is essential[14]. In this case the background metric $g_{\mu\nu}$ generated by the mass distribution is assumed to be a slowly varying function of the time on the inverse temperature scale[14]. More precisely, we assume

$$\frac{\partial g_{\mu\nu}}{\partial t} \ll \frac{g_{\mu\nu}}{\beta} \, . \tag{2}$$

The fluctuation of the metric is due to the energy momentum tensor fluctuations. If we ignore the energy fluctuations, (equilibrium state) , we can assume that the metric is static or slow

varying with respected to the time. We neglect the influence of the matter and radiation and also assume that their interactions with DE are small and serve only to provide a heat bath at the temperature T. The boundary is a timelike tube which is periodically identified in the imaginary time direction with the period β. The necessary path integral for the construction of the partition function is taken over asymptotically vanishing fields which are periodic in the imaginary time t with the period β. Thus, the functional integration has been assumed with the periodicity in the imaginary time and the asymptotic flatness of the metric fields. As we know that in any static or slowly time varying spacetimes the Ricci scalar has no dynamics (time dependency) or it's variations with respect to the time is negligible,but only in these models we can write a suitable explicit expression for Green function. On the other hand the static spacetime implies that the ensemble of Bosons remains at a definite constant temperature in an equilibrium or a quasi equilibrium state of the system .If we are far away from a static space times, all equations has only a perturbative solution and it's applications of an expanding universe is doubtful. There is only some few published works in this topics and finally leads to a stochastic statistical field theory[20]. [1] The topology of this manifold is considered to be a $S^1 \times M^3$ and also we can take M^3 manifold without any boundary[17] .

III: THERMODYNAMIC POTENTIAL

In this section we will analyze the scalar field model with a conserved charge.
The Lagrangian of the model is:

$$S_m = -(\frac{1}{2}) \int d^4x \sqrt{g(x)} \Phi^*(x)(-\Box_x + m^2 + \xi R)\Phi(x) \tag{3}$$

Where $\Phi = (\phi_1, \phi_2)$ is a doublet of the real fields. The action written in the terms of real fields will be:

$$S_m = -(\frac{1}{2}) \int d^4x \sqrt{g(x)} \phi^a(x)(-\Box_x + m^2 + \xi R)\phi_a(x) \tag{4}$$

The total action of the system "$matter + gravity$" is:

$$S_{tot} = S_g + S_m \tag{5}$$

Now we can write the effective action at finite temperature as:

$$L_{eff}(\beta) = \tilde{L}_g - \omega(\beta, \mu, R) \tag{6}$$

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[1] Stochastic semi classical gravity in the 1990s is a theory naturally evolved from semi classical gravity in the 1970s and 1980s. In stochastic semiclassical gravity the main object of interest are the noise kernel, the vacuum expectation value of the (operator-valued) stress-energy bi-tensor, and the centerpieces being the (semiclassical) Einstein-Langevin equation. It also brings out the open system concepts and the statistical and stochastic contents of the theory such as dissipation, fluctuations, noise and decoherence. Hu[20] has been described the applications of the stochastic gravity to the back reaction problems in the cosmology and the black-hole physics. Further discussions of the ideas and the ongoing research topics can be found in [21,22,23]

where $\tilde{L}_g$ is

$$\tilde{L}_g = L_g - \frac{i}{2} \int_{m^2}^{\infty} dm^2 tr G_{SD}(x, \acute{x})$$

The symbol $tr(...)$ is determined as:

$$tr(...) = \sum_{n \neq 0} \int \frac{d^3 k}{(2\pi)^3} ... \tag{7}$$

and $\omega(\beta, \mu, R)$ is the density of grand thermodynamic potential.
The result (6) may be obtained with the momentum space representation for the Green's function of a Boson .
In the momentum space representation, the expression for L_{eff} is splited into two parts:

$$L_{eff} = -(\frac{i}{2}) \int_{m^2}^{\infty} dm^2 tr G(x, \acute{x}) - \omega(\beta, \mu, R) \tag{8}$$

The potential $\omega(\beta, R)$ is:

$$\omega(\beta, R) = -(\frac{1}{2}) tr \int_{m^2}^{\infty} dm^2 \sum_{j=0}^{2} \sum_{n=0}^{\infty} \gamma_j(R)(-\frac{\partial}{\partial m^2})^j \times \int \frac{d^3 k}{(2\pi)^3} (\omega_n^2 + \epsilon^2)^{-1} \tag{9}$$

$$= (\frac{1}{2}) \sum_{j=0}^{2} \gamma_j(R)(-\frac{\partial}{\partial m^2})^j tr \ln(\omega_n^2 + \epsilon^2)$$

where $\omega_n = \frac{2\pi n}{\beta}$ and the geometrical coefficients $\gamma_j(R)$ can be represented in terms of scalar R and coupling constant ξ as:

$$\gamma_0(R) = 1$$
$$\gamma_1(R) = (\frac{1}{6} - \xi)R$$
$$\gamma_2(R) = -\frac{1}{180}R_{\mu\nu}R^{\mu\nu} + \frac{1}{180}R_{\mu\nu\sigma\tau}R^{\mu\nu\sigma\tau} + \frac{1}{6}(\frac{1}{5} - \xi)R_{;\mu}^{\mu}$$

For introducing the chemical potential we will change the Matsubara frequencies[24] $\omega_n \rightarrow \omega_n + \mu$ and then thermodynamic potential will be $\omega(\beta, \mu, R)$.
Since both positive and negative frequencies are summed, we will get:

$$tr \ln(\omega_n^2 + \epsilon^2) \rightarrow tr \ln[(\omega_n + \mu)^2 + \epsilon^2]$$
$$= tr\{\ln[\omega_n^2 + (\epsilon - \mu)^2] + \ln[\omega_n^2 + (\epsilon + \mu)^2]\} \tag{10}$$

After doing required mode's summation in (9) we will have:

$$\omega(\beta, \mu, R) = \omega_-(\beta, \mu, R) + \omega_+(\beta, \mu, R) \tag{11}$$

Where:

$$\omega_-(\beta, \mu, R) = (\frac{1}{\beta}) \sum_{j=0}^{2} \gamma_j(R)(-\frac{\partial}{\partial m^2})^j \ln(1 - \exp[-\beta(\epsilon - \mu)]) \tag{12}$$

And:

$$\omega_+(\beta, \mu, R) = (\frac{1}{\beta}) \sum_{j=0}^{2} \gamma_j(R)(-\frac{\partial}{\partial m^2})^j \ln(1 - z \exp[-\beta(\epsilon - \mu)]) \tag{13}$$

So the density of grand thermodynamic potential is the series:

$$\omega(\beta, \mu, R) = \sum_{j=0}^{2} \gamma_j(R) b_j(\beta m, z) \tag{14}$$

Where:

$$b_0(\beta m, z) = (\frac{1}{\beta}) \ln(1 - z \exp(-\beta\epsilon));$$

$$b_j(\beta m, z) = (-\frac{\partial}{\partial m^2})^j b_0(\beta m, z) \tag{15}$$

and the fugacity is $z = \exp(\beta\mu)$.

IV: STATISTICS AND THERMODYNAMICS OF AN IDEAL (NON INTERACTIVE) BOSE GAS

The Bose distribution function as the derivative of the grand thermodynamic potential is given by:

$$n_{\vec{k}} = -\frac{\partial \omega_{\vec{k}}(\beta, \mu, R)}{\partial \mu} \tag{16}$$

For occupation numbers with momentum $\vec{k}$ we can obtain:

$$n_{\vec{k}} = \frac{1}{(z^{-1}e^{\beta\epsilon_{\vec{k}}} - 1)} B(\beta, R) \tag{17}$$

Where the function $B(\beta, R)$ is described by the formula:

$$B(\beta, R) = 1 + \gamma_1(R)\frac{\beta}{2\varepsilon_{\vec{k}}}[1 - (1 - z \exp^{-\beta\varepsilon_{\vec{k}}})^{-1}] + ... \tag{18}$$

The function $B(\beta, R)$ depends on the curvature, the temperature and the energy of the Boson[18].
Studying the thermodynamic properties of the Bose gases we will start with the equation:

$$\omega(\beta, \mu, R) = -(\frac{1}{\beta}) \sum_{j=0}^{2} \gamma_j(R)(-\frac{\partial}{\partial m^2})^j \ln(1 - z \exp[-\beta\epsilon]) \tag{19}$$

In the non-relativistic limit for the particle energy $\epsilon = \frac{\vec{k}^2}{2m}$ we can derive from (19) the equation:

$$\omega(\beta, \mu, R) = \sum_{j=0}^{2} \gamma_j(R) g_{5/2}(z)(-\frac{\partial}{\partial m^2})^j \lambda^{-3} \tag{20}$$

where $\lambda = (2\pi/mT)^{1/2}$ is a wavelength of the particle, and the function $g_{5/2}(z)$ has the following form:

$$g_{5/2}(z) = \sum_{l=1}^{\infty} \frac{z^l}{l^{5/2}} - \frac{4}{\sqrt{\pi}} \int_0^{\infty} x^2 \ln(1 - z \exp(-x^2)) dx$$

The average number of the particles in a certain momentum state $\overrightarrow{k}$ is obtained as the derivative:

$$< n_{\overrightarrow{k}} >= -\frac{\partial}{\partial \mu} \omega(\beta, \mu, R) = \sum_{j=0}^{2} \gamma_j(R) g_{5/2}(z)(-\frac{\partial}{\partial m^2})^j (z^{-1} \exp(\beta\epsilon) - 1)$$

The density of the particles is:

$$n = \lambda^{-3}[1 - \gamma_1(R)(3/4m^2) - \gamma_2(R)(3/16m^4)]g_{3/2}(z) + n_0 \tag{21}$$

Where the new function $g_{3/2}(z)$ is:

$$g_{3/2}(z) = z\frac{\partial}{\partial z}g_{5/2}(z)$$

and

$$n_0 = \frac{z}{1 - z}$$

is the average number of the particles with zero momentum. The functions $g_{3/2}(z)$ and $g_{5/2}(z)$ are special cases of a more general class of functions:

$$g_n(z) = \sum_{k=1}^{\infty} z^k/k^n \tag{22}$$

In a more simple form the equation (21) may be written as:

$$(n - n_0)\lambda^3 = g_{3/2}(z, R) \tag{23}$$

Where:

$$g_{3/2}(z, R) = [1 - \alpha\frac{R}{m^2} + ...]g_{3/2}(z) \tag{24}$$

is a function which depends on the curvature, and α is a numerical parameter.
The equation (23) connects four values: fugacity, temperature, density of the particles and curvature.

V: CALCULATING THE DENSITY OF STATES FOR EOS $p = -\rho$

The density of grand thermodynamic potential is :

$$\omega(\beta, \mu, R) = \frac{\Omega}{V} - \frac{1}{\beta} \sum_{j=0}^{2} \gamma_j(R)(-\frac{\partial}{\partial m^2})^j \sum_{\epsilon} \ln(1 - z exp(-\beta\epsilon)), \tag{25}$$

where $\beta = 1/T$.

We can write the sum over the energy states as an integration over energy values by considering it's occupation number function (Degeneracy of the energy levels) $g(\epsilon)$ where:

$$g(\epsilon)d\epsilon = \frac{V}{(2\pi)^3}d^3k$$

We know that for photons, phonons and non-relativistic particles with spin s this function is(in Geometrical units where: $\hbar = c = 1$) [11] :

$$g(\epsilon) = \frac{V\epsilon^2}{(2\pi)}(photons)$$

$$g(\epsilon) = \frac{9N\epsilon^2}{\omega_D^3}(phonons)$$

$$g(\epsilon) = \frac{(2s+1)Vm^{3/2}\epsilon^{1/2}}{\sqrt{2}\pi^2}(NonrelativisticBosegas)$$

We now review some relations between the Grand Thermodynamic Potential of the system $\Omega(\beta, z, V)$, the Helmholtz Free energy (F), the entropy (S)and the pressure (P) [11]:

$$F = U - TS$$
$$U = \Omega + \mu N + TS$$
$$PV = -\Omega$$
$$S = -\frac{\partial \Omega}{\partial T}$$
$$\Omega = \beta(Ln(Z(\mu, T, N)))$$
$$U = -(\frac{\partial Ln(Z(\mu, T, N))}{\partial \beta}),$$

where in these relations z is called fugacity :

$$z = e^{\beta\mu}$$

Since now we named the density of the Grand potential of the system as ω then we can write next equations for the density (per volume) of them as:

$$f = \rho - Ts \tag{26}$$
$$\rho = \omega + \mu n + Ts \tag{27}$$
$$P = -\omega \tag{28}$$

The main goal of this paper is deriving degeneracy function per volume for an EOS $p = -\rho$, and then if it is the well behavior function, we can fit it with an interaction inside a Boson gas, then the generation function of both systems will become the same. Our result for

the dispersion relation has been shown that there is the k^3 dependence of the energy. This dependence can be the main effective term of the interaction in a definite density and temperature, and also like the logarithmic dependence of the dispersion relation strength quark matter (SQM); the k^3 term can be dominant in the some regions[19].

Remembering previous relations together with EOS $p = -\rho$ we lead to the next first order partial differential equation for ω.

$$\mu\frac{\partial\omega}{\partial\mu} = \beta\frac{\partial\omega}{\partial\beta} \tag{29}$$

now we note here that the general solution for above equation in which

$$\omega(\mu, \beta) = \sum_a c_a(\mu\beta)^a \tag{30}$$

we have a simple boundary condition which is concluded from thermodynamics: the entropy of a system must be vanish at absolute zero temperature β. that is we must have:

$$\lim_{\beta\to\infty} (\beta^2\frac{\partial\omega}{\partial\beta}) = 0$$

Thus we must limit ourselves only to $a < -1$. But the coefficients remain undetermined. Thus we take another endeavor to solve this equation. We substitute (25) in (29) and take derivatives and equalize from both sides all terms which have the same order of m^2 (i.e.; the coefficients of $(-\frac{\partial}{\partial m^2})^j, j = 0, 1, 2$ We have:

$$\int_0^\infty \frac{\epsilon g(\epsilon)}{e^{-\beta(\mu-\epsilon)} - 1}d\epsilon - \frac{1}{\beta}\int_0^\infty \ln(1 - ze^{-\beta\epsilon})g(\epsilon)d\epsilon \tag{31}$$

We assume that $g(\epsilon) = \frac{d\sigma(\epsilon)}{d\epsilon}$ and also we suppose that $\sigma(\epsilon)$ is a well behavior function of energy ϵ at the ground state energy level $\epsilon = 0$. After integrating part by part and expanding both sides in terms of z $(0 < z < 1)$ and set equal all terms of same z power we have:

$$\int_0^\infty e^{-\beta n\epsilon}[\sigma(\epsilon) - \epsilon\frac{d\sigma}{d\epsilon}]d\epsilon = \frac{\sigma(0)}{\beta n}, n \geq 1 \tag{32}$$

We can define Laplace transformation of the function $\sigma(\epsilon)$ as a function of parameter $s = \beta n$ by:

$$\Sigma(s) = \int_0^\infty e^{-s\epsilon}\sigma(\epsilon)d\epsilon \tag{33}$$

We arrive at the next differential equation for the transformed function:

$$\Sigma(s) + \frac{\partial}{\partial s}(s\Sigma(s) - \sigma(0)) = \frac{\sigma(0)}{s} \tag{34}$$

Which has the solution:

$$\Sigma(s) = \frac{\sigma(0)}{s} + \frac{C_1}{s^2} \tag{35}$$

In this function C_1 is an unknown constant that can be determined using initial conditions. But we take it as a constant and find it's value some later. Because we solve the partial differential equation (29), then the result can be a function of other quantities like the mass of particles. The mass is not an independent thermodynamic variable. If we assume the mass as a thermodynamic variable, then the derivative of mass with respected to the density, the temperature or the chemical potential must be occurred in the fundamental thermodynamic relations [19].

By taking inverse Laplace transformation from (35)we obtain:

$$\sigma(\epsilon) = \sigma(0) + C_1\epsilon \qquad (36)$$

We mention here that for a Boson's system the chemical potential is always negative namely $-\infty < \mu \leq 0$. Since $g(\epsilon) = \frac{d\sigma(\epsilon)}{d\epsilon}$ we finally obtain:

$$g(\epsilon) = C_1 \qquad (37)$$

Substituting (37) in definition of $g(\epsilon)$ [2] we lead to the following dispersion relation for DE:

$$k^3 = 6\pi^2 C_1\epsilon - 6\pi^2 C_2 \qquad (38)$$

We know that this is not a dispersion relation for a non interactive matter field.

But there is no simple system which has such distribution. If we assume that the energy levels of such Bosonic system must be positive and the momentum of any particle also be positive, then the energy levels of this system must satisfy the following inequality:

$$\epsilon \geq \frac{C_2}{C_1} \qquad (39)$$

Without any dismay from loss of generality we can take $C_2 = 0$ .With this choice we accept that our ground state energy level is located at $\epsilon = 0$. Another pickup is tagging this energy level a non zero value. This situation could be solved by a shift of any energy value up to this value.

Substituting (37) in (25) and making use from:

$$Poly\log(2, z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2},$$

finally we obtain next function for the density of grand thermodynamic potential:

$$\omega(\mu, R, \beta) = \frac{1}{\beta^2} Polylog(2, z) \sum_{j=0}^{2} a_j(R)\left(-\frac{\partial}{\partial m^2}\right)^j C_1 \qquad (40)$$

Now we calculate the density of the average of the energy .We can label it as ρ_Λ which is:

$$\rho_\Lambda = \frac{\bar{E}}{V} = \int_0^{\infty} \frac{g(\epsilon)\epsilon}{e^{-\beta(\mu-\epsilon)} - 1} d\epsilon$$

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[2] Note that V is disappeared since we calculate density.

After substituting (37) in it and doing the integration we obtain:

$$\rho_\Lambda = \frac{C_1}{\beta^2} Polylog(2, z) \tag{41}$$

The mean value of the number of the particles in the unit volume of the system is:

$$n = \frac{\bar{N}}{V} = \int_0^\infty \frac{g(\epsilon)}{e^{-\beta(\mu-\epsilon)} - 1} d\epsilon = -\frac{C_1}{\beta} \ln(1 - z) \tag{42}$$

Since $0 < z < 1$ the value of n is not negative. Using $s = \frac{S}{V} = -\frac{\partial \omega}{\partial T}$ we have:

$$s = -\frac{2}{\beta} Poly \log(2, z) \sum_{j=0}^2 a_j(R)(-\frac{\partial}{\partial m^2})^j C_1 \tag{43}$$

Because the mass dependence of $g(\epsilon)$ is undetermined , so we can not conclude strongly about the sign of entropy. It can be negative or positive [14].

Negative entropy is problematic if we accept that the entropy is in an the association with the measure of the number of microstates in statistical mechanics. The intuition of the statistical mechanics requires that the entropy of all physical components have to be positive. Besides if we consider the universe as a thermodynamic system, the total entropy of the universe including DE and DM should satisfy the second law of thermodynamics. The GSL for phantom and non-phantom DE has been explored in [12]. It was found that the GSL can be protected in the universe with DE. The GSL of the universe with DE has been investigated in [13,25,26,27] as well. [3] ,and also the equation of state of the DE is uniquely determined and the phantom entropy is negative [13]. For the phantom,when we running to the negative entropy problem, the GSL is violated.[4] Considering the apparent horizon as the physical boundary of the universe, it was found that both the temperature and the entropy can be positive for the DE, including phantom.In the general framework that we discussed, where the cosmological metric has a slow varying time dependency, we can assume that the time contiguity of the volume and the temperature is venial and for this reason we have obtained a negative entropy. We must note here that if in an ensemble of particles the volume and temperature were non static, the statistical behavior of this system is not in the context of the equilibrium thermodynamics.[5]

VI: CONCLUSIONS

We have derived a grand canonical partition function for a DE's like EOS. The thermodynamic equations are generally expressed in terms of three variables: the temperature T ,

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[3] We may think the DE temperature is equal or proportional to the horizon temperature T_H.

[4] It is more realistic to consider that the physical volume and the temperature of the universe are related, since in the general situation they both depend on the scale factor a(t).It was found that the apparent horizon is a good boundary for keeping thermodynamics' laws [13].

[5] As in a FRW standard model of cosmology, since our background metric is not static and thereupon the apparent horizon is time variable, the construction of a local quantum statistical and counting all microstates must affect the negative entropy and we can deduce a statistical proof for non negativity and non violation of GSL like Gong and collaborations proof [6].

chemical potential μ associated with a conserved particle number N, and scalar curvature R. We also reviewed some known results for an ideal Bose gas in curved spacetime. Comparing our results with an ideal gas, we have concluded that there is a significance generic difference between the statistical distribution of negative pressure matters and ideal gases. Specifically , the dispersion relation for a dark energy like EOS is non quadratic (in contradiction with a non interactive ideal Bose gas)and comparing this dispersion function with the statistical model of SQM we observe that also in SQM systems this dispersion relation is valid in some physically accessible region of the system[19].

Also we have derived the mean value of the energy of the system in terms of the curvature . We have shown that if a matter field (Boson) obeys an EOS like DE, and in cosmological language violates the well known energy conditions in General relativity , the degeneracy function of this gas must be an implicit constant function of the energy. This function implicitly is dependent on the mass of the particles which is not a thermodynamic quantity and remains as an undetermined function in this formula.

The derived thermodynamic equations has been shown that the entropy may be positive or negative. As it was stated in [6], this negativity of the entropy is hidden behind this significance assumption that the volume and the temperature are independent. But we know that in an expanding FRW universe both of them are functions of time. Hence if we assume that our metric is quasi static or has a negligible dynamics, any phantom field of DE with an intelligible statistical testimony has a negative entropy. If we want to generalize our argument to a realistic cosmological case, we must work in a non equilibrium statistical regime.

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Some notes about the density of state function for a negative pressure matter

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The main goal of this paper is deriving Density of states $g(\epsilon)$ (degeneracy function) per volume for an equation of state (EOS) $p = -\rho$ (we called it dark energy(DE)). We concluded that thermodynamic quantities such as pressure and energy density are simple functions of Temperature , fugacity, curvature and mass of Bosons. We found the corresponding entropy and also our work exposure the statistical pedigree of some claims about scalar fields which presenting negative pressure.

I: INTRODUCTION

In recent years some works have been done on the thermal properties of negative pressure matters specially on the dark energy . The main assumption of all these models is that the DE is a thermalized ensemble at a certain temperature with a specific amount of entropy[1]. It would be usable to take temperature as an intrinsic characteristics of DE, namely a barotropic matter with an EOS as $p = f(\rho)$. This fluid must not violate energy conditions. Also we can take both dark matter (DM) and dark energy as different types of the same matter field with a phase transition in cosmological distances[2]. There is not any reasonable statistical description in negative pressure matters .In some of recent papers, thermal properties of dark energy have been discussed in terms of the assumption that the dark energy substance is a thermalized ensemble at a certain temperature with an associated thermo dynamical entropy[1]. It is usually assumed that this temperature is an intrinsic property of DE rather than the temperature of the heat bath fixed by surrounding matter. Checking the second law of thermodynamics in the context of the black hole physics is another important problem.Also the generalized second law (GSL) must be satisfied by any physical description of the DE even in the new scenarios of the modified gravity in the non relativistic , diffeomorphism broken modeles such as Hořava proposal[29].If we denote that S_{tot} is the total entropy of a system,then GSL guaranties that $\dot{S}_{tot} \geq 0$ for all times.Recently Mubasher Jamil , et.al showed that[26] in an interactive model for DE , DM and radiation "*the generalized second law is always and generally valid, independently of the specific interaction form of the fluid's equation-of-state parameters and of the background geometry.*".The ancestry of GSL backs to the Unruh and Wald classical work[28]and Bekenstein works on the black holes[31].There exists a recondite relation between the holographic dark energy model and generalized second law of thermodynamics as was pointed out in [27] and also in in quintom dominated universe [30].Holographic scenario for DE is so popular and was investigated by authors both in the context of the usual FRWL cosmology and also in

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the modified ones and in a braneworld picture of universe[25,32,33,36].There is a delicate relation between Bulk brane interaction , holographic dark energy [34], Gauss Bonnet dark energy models[35,38],Stringy inspired tachyon model[37],phantom like regime [39],Observational constraints on holographic dark energy[40], Holographic Chaplygin gas model[41], Holographic Chaplygin DGP Cosmologies[42]and even Holographic Modified Gravity[43].

Some years ago Kulikov and Pronin [3, 15] constructed a simple formulation of a local quantum statistics of Bosonic field in a curved background. Indeed they explained a simple expression for a grand canonical thermodynamic potential density [4] which locally describes all thermo chemical properties of gases with a curvature dependent parts [5].Later The high-temperature expansion of the grand thermodynamic potential of non conformably invariant spin-0 and spin-1/2 gases in an arbitrary static spacetime with their boundary is calculated by method conforming from Kirsten [16].

Through this report, we have worked on the statistics of a negative matter with EOS like DE. At first we have obtained the degeneracy distribution function for massive Bosons in this grand ensemble and it has been shown that the density of states for such exotic matter is a constant function of energy . We have summed over all energy states once more and obtained a closed form for grand canonical thermodynamic potential density and then we have inferred the expressions for the density of entropy and other important functions. We have also show that these thermodynamic equations are generally expressed in terms of fugacity , Ricci scalar and consequently on the temperature and chemical potential. The chemical potential has been associated with a conserved Boson number . We have been stated that from a statistical point of view DE violate energy conditions companionship to the claims stated in the recent papers[6].To shunning from this undesirable result we must take volume and temperature as two dependence variables as Gong and collaborations showed in [6].

II: LAGRANGIAN OF THE MODEL AND SOME REMARKS ABOUT THE BACKGROUND METRIC

We begin from Lagrangian of a massive scalar field coupled to gravity which we assume that is non negative. Consider the action[8]

$$S = \int \sqrt{-g}d^4x(\frac{1}{2}\phi^{;\mu}\phi_{;\mu} - \frac{1}{2}(m^2 + \xi R)\phi^2) \tag{1}$$

where R is the scalar curvature and we assume that the metric of space time $g_{\mu\nu}$ is static or slowly time variable such that we can take the Ricci scalar, time independent. Spherical symmetry is not essential in this formalism, because we use the general form for the metric, and only we impose a restriction on the time variation of the metric. The only essential assumption is on the surface tension energy portion in EOS. In the case of bulk matter we ignore the surface effects and all of the thermodynamics quantities are expressed per unit volume. Considering the equilibrium assumption , a static metric is essential[14]. In this case the background metric $g_{\mu\nu}$ generated by the mass distribution is assumed to be a slowly varying function of the time on the inverse temperature scale[14]. More precisely, we assume

$$\frac{\partial g_{\mu\nu}}{\partial t} \ll \frac{g_{\mu\nu}}{\beta} . \tag{2}$$

The fluctuation of the metric is due to the energy momentum tensor fluctuations. If we ignore the energy fluctuations, (equilibrium state) , we can assume that the metric is static or slow varying with respected to the time. We neglect the influence of the matter and radiation and also assume that their interaction with DE are small and serve only to provide a heat bath at the temperature T. The boundary is a timelike tube which is periodically identified in the imaginary time direction with the period β. The necessary path integral for the construction of the partition function is taken over asymptotically vanishing fields which are periodic in imaginary time t with period β. Thus, the functional integration has been assumed with the periodicity in imaginary time and the asymptotic flatness of the metric fields. As we know that in any static or slowly time varying spacetimes the Ricci scalar has no dynamics (time dependency) or it's variations with respect to the time is negligible,but only in these models we can write a suitable explicit expression for Green function. On the other hand the static spacetime implies that the ensemble of Bosons remains at a definite constant temperature in an equilibrium or a quasi equilibrium state of the system .If we are far away from a static space times, all equations has only a perturbative solution and it's applications of an expanding universe is doubtful. There is only some few published works in this topics and finally leads to a stochastic statistical field theory[20]. [1] The topology of this manifold is considered to be a $S^1 \times M^3$ and also we can take M^3 manifold without any boundary[17] .

III: THERMODYNAMIC POTENTIAL

In this section we will analyze the scalar field model with a conserved charge.
The Lagrangian of the model is:

$$S_m = -(\frac{1}{2}) \int d^4x \sqrt{g(x)} \Phi^*(x)(-\Box_x + m^2 + \xi R)\Phi(x) \tag{3}$$

Where $\Phi = (\phi_1, \phi_2)$ is a doublet of the real fields. The action written in the terms of real fields will be:

$$S_m = -(\frac{1}{2}) \int d^4x \sqrt{g(x)} \phi^a(x)(-\Box_x + m^2 + \xi R)\phi_a(x) \tag{4}$$

The total action of the system "$matter + gravity$" is:

$$S_{tot} = S_g + S_m \tag{5}$$

Now we can write the effective action at finite temperature as:

$$L_{eff}(\beta) = \tilde{L}_g - \omega(\beta, \mu, R) \tag{6}$$

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[1] Stochastic semi classical gravity in the 1990s is a theory naturally evolved from semi classical gravity in the 1970s and 1980s. In stochastic semiclassical gravity the main object of interest are the noise kernel, the vacuum expectation value of the (operator-valued) stress-energy bi-tensor, and the centerpieces being the (semiclassical) Einstein-Langevin equation. It also brings out the open system concepts and the statistical and stochastic contents of the theory such as dissipation, fluctuations, noise and decoherence. Hu[20] has been described the applications of the stochastic gravity to the back reaction problems in the cosmology and the black-hole physics. Further discussions of the ideas and the ongoing research topics can be found in [21,22,23]

where $\tilde{L}_g$ is

$$\tilde{L}_g = L_g - \frac{i}{2} \int_{m^2}^{\infty} dm^2 tr G_{SD}(x, \acute{x})$$

The symbol $tr(...)$ is determined as:

$$tr(...) = \sum_{n \neq 0} \int \frac{d^3 k}{(2\pi)^3} ... \tag{7}$$

and $\omega(\beta, \mu, R)$ is the density of grand thermodynamic potential.
The result (6) may be obtained with the momentum space representation for the Green's function of a Boson .
In the momentum space representation, the expression for L_{eff} is splited into two parts:

$$L_{eff} = -(\frac{i}{2}) \int_{m^2}^{\infty} dm^2 tr G(x, \acute{x}) - \omega(\beta, \mu, R) \tag{8}$$

The potential $\omega(\beta, R)$ is:

$$\omega(\beta, R) = -(\frac{1}{2}) tr \int_{m^2}^{\infty} dm^2 \sum_{j=0}^{2} \sum_{n=0}^{\infty} \gamma_j(R) (-\frac{\partial}{\partial m^2})^j \times \int \frac{d^3 k}{(2\pi)^3} (\omega_n^2 + \epsilon^2)^{-1} \tag{9}$$

$$= (\frac{1}{2}) \sum_{j=0}^{2} \gamma_j(R) (-\frac{\partial}{\partial m^2})^j tr \ln(\omega_n^2 + \epsilon^2)$$

where $\omega_n = \frac{2\pi n}{\beta}$ and the geometrical coefficients $\gamma_j(R)$ can be represented in terms of scalar R and coupling constant ξ as:

$$\gamma_0(R) = 1$$
$$\gamma_1(R) = (\frac{1}{6} - \xi)R$$
$$\gamma_2(R) = -\frac{1}{180} R_{\mu\nu} R^{\mu\nu} + \frac{1}{180} R_{\mu\nu\sigma\tau} R^{\mu\nu\sigma\tau} + \frac{1}{6}(\frac{1}{5} - \xi)R_{;\mu}^{\mu}$$

For introducing the chemical potential we will change the Matsubara frequencies[24] $\omega_n \rightarrow \omega_n + \mu$ and then thermodynamic potential will be $\omega(\beta, \mu, R)$.
Since both positive and negative frequencies are summed, we will get:

$$tr \ln(\omega_n^2 + \epsilon^2) \rightarrow tr \ln[(\omega_n + \mu)^2 + \epsilon^2]$$
$$= tr\{\ln[\omega_n^2 + (\epsilon - \mu)^2] + \ln[\omega_n^2 + (\epsilon + \mu)^2]\} \tag{10}$$

After doing required mode's summation in (9) we get:

$$\omega(\beta, \mu, R) = \omega_-(\beta, \mu, R) + \omega_+(\beta, \mu, R) \tag{11}$$

Where:

$$\omega_-(\beta, \mu, R) = (\frac{1}{\beta}) \sum_{j=0}^{2} \gamma_j(R) (-\frac{\partial}{\partial m^2})^j \ln(1 - \exp[-\beta(\epsilon - \mu)]) \tag{12}$$

And:

$$\omega_+(\beta, \mu, R) = (\frac{1}{\beta}) \sum_{j=0}^{2} \gamma_j(R)(-\frac{\partial}{\partial m^2})^j \ln(1 - z\exp[-\beta(\epsilon - \mu)]) \tag{13}$$

So the density of grand thermodynamic potential is the series:

$$\omega(\beta, \mu, R) = \sum_{j=0}^{2} \gamma_j(R)b_j(\beta m, z) \tag{14}$$

Where:

$$b_0(\beta m, z) = (\frac{1}{\beta})\ln(1 - z\exp(-\beta\epsilon));$$

$$b_j(\beta m, z) = (-\frac{\partial}{\partial m^2})^j b_0(\beta m, z) \tag{15}$$

and the fugacity is $z = \exp(\beta\mu)$.

IV: STATISTICS AND THERMODYNAMICS OF AN IDEAL (NON INTERACTIVE) BOSE GAS

The Bose distribution function as the derivative of the grand thermodynamic potential is given by:

$$n_{\vec{k}} = -\frac{\partial \omega_{\vec{k}}(\beta, \mu, R)}{\partial \mu} \tag{16}$$

For occupation numbers with momentum $\vec{k}$ we can obtain:

$$n_{\vec{k}} = \frac{1}{(z^{-1}e^{\beta\epsilon_{\vec{k}}} - 1)}B(\beta, R) \tag{17}$$

Where the function $B(\beta, R)$ is described by the formula:

$$B(\beta, R) = 1 + \gamma_1(R)\frac{\beta}{2\varepsilon_{\vec{k}}}[1 - (1 - z\exp^{-\beta\varepsilon_{\vec{k}}})^{-1}] + ... \tag{18}$$

The function $B(\beta, R)$ depends on the curvature, the temperature and the energy of the Boson[18].
Studying the thermodynamic properties of the Bose gases we will start with the equation:

$$\omega(\beta, \mu, R) = -(\frac{1}{\beta}) \sum_{j=0}^{2} \gamma_j(R)(-\frac{\partial}{\partial m^2})^j \ln(1 - z\exp[-\beta\epsilon]) \tag{19}$$

In the non-relativistic limit for the particle energy $\epsilon = \frac{\vec{k}^2}{2m}$ we can derive from (19) the equation:

$$\omega(\beta, \mu, R) = \sum_{j=0}^{2} \gamma_j(R)g_{5/2}(z)(-\frac{\partial}{\partial m^2})^j \lambda^{-3} \tag{20}$$

where $\lambda = (2\pi/mT)^{1/2}$ is a wavelength of the particle, and the function $g_{5/2}(z)$ has the following form:

$$g_{5/2}(z) = \sum_{l=1}^{\infty} \frac{z^l}{l^{5/2}} = -\frac{4}{\sqrt{\pi}} \int_0^{\infty} x^2 \ln(1 - z\exp(-x^2))dx$$

The average number of the particles in a certain momentum state $\overrightarrow{k}$ is obtained as the derivative:

$$< n_{\overrightarrow{k}} > = -\frac{\partial}{\partial \mu}\omega(\beta, \mu, R) = \sum_{j=0}^{2} \gamma_j(R)g_{5/2}(z)(-\frac{\partial}{\partial m^2})^j(z^{-1}\exp(\beta\epsilon) - 1)$$

The density of the particles is:

$$n = \lambda^{-3}[1 - \gamma_1(R)(3/4m^2) - \gamma_2(R)(3/16m^4)]g_{3/2}(z) + n_0 \tag{21}$$

Where the new function $g_{3/2}(z)$ is:

$$g_{3/2}(z) = z\frac{\partial}{\partial z}g_{5/2}(z)$$

and

$$n_0 = \frac{z}{1-z}$$

is the average number of the particles with zero momentum. The functions $g_{3/2}(z)$ and $g_{5/2}(z)$ are special cases of a more general class of functions:

$$g_n(z) = \sum_{k=1}^{\infty} z^k/k^n \tag{22}$$

In a more simple form the equation (21) may be written as:

$$(n - n_0)\lambda^3 = g_{3/2}(z, R) \tag{23}$$

Where:

$$g_{3/2}(z, R) = [1 - \alpha\frac{R}{m^2} + ...]g_{3/2}(z) \tag{24}$$

is a function which depends on the curvature, and α is a numerical parameter.
The equation (23) connects four values: fugacity, temperature, density of the particles and curvature.

V: CALCULATING THE DENSITY OF STATES FOR EOS $p = -\rho$

The density of grand thermodynamic potential is :

$$\omega(\beta,\mu,R) = \frac{\Omega}{V} = -\frac{1}{\beta}\sum_{j=0}^{2}\gamma_j(R)(-\frac{\partial}{\partial m^2})^j \sum_\epsilon \ln(1 - z\exp(-\beta\epsilon)), \tag{25}$$

where $\beta = 1/T$.

We can write the sum over energy states as an integration over energy values by considering it's occupation number function (Degeneracy of energy levels) $g(\epsilon)$ where:

$$g(\epsilon)d\epsilon = \frac{V}{(2\pi)^3}d^3k$$

We know that for photons, phonons and non-relativistic particles with spin s this function is(in Geometrical units where: $\hbar = c = 1$) [11] :

$$g(\epsilon) = \frac{V\epsilon^2}{(2\pi)}(photons)$$

$$g(\epsilon) = \frac{9N\epsilon^2}{\omega_D^3}(phonons)$$

$$g(\epsilon) = \frac{(2s+1)Vm^{3/2}\epsilon^{1/2}}{\sqrt{2}\pi^2}(Nonrelativistic Bosegas)$$

We now review some relations between the Grand Thermodynamic Potential of the system $\Omega(\beta, z, V)$ and the Helmholtz Free energy of system(F), entropy (S)and pressure (P) [11]:

$$F = U - TS$$
$$U = \Omega + \mu N + TS$$
$$PV = -\Omega$$
$$S = -\frac{\partial\Omega}{\partial T}$$
$$\Omega = \beta(Ln(Z(\mu, T, N)))$$
$$U = -(\frac{\partial Ln(Z(\mu, T, N))}{\partial\beta}),$$

where in these relations z is called fugacity :

$$z = e^{\beta\mu}$$

Since now we named the density of the Grand potential of the system as ω then we can write next equations for the density (per volume) of them as:

$$f = \rho - Ts \tag{26}$$
$$\rho = \omega + \mu n + Ts \tag{27}$$
$$P = -\omega \tag{28}$$

The main goal of this paper is deriving degeneracy function per volume for an EOS $p = -\rho$, and then if it is the well behavior function, we will fit it with an interaction inside a Boson gas, then the generation function of both systems will become the same. Our result for

dispersion relation shows that there is the k^3 dependence of the energy. This dependence can be the main effective term of the interaction in a definite density and temperature and like the logarithmic dependence of the dispersion relation strength quark matter (SQM)(in the some regions) the k^3 term can be dominant [19].

Remembering previous relations together with EOS $p = -\rho$ we lead to the next first order partial differential equation for ω.

$$\mu\frac{\partial\omega}{\partial\mu} = \beta\frac{\partial\omega}{\partial\beta} \tag{29}$$

now we note here that the general solution for above equation is:

$$\omega(\mu,\beta) = \sum_a c_a(\mu\beta)^a \tag{30}$$

we have a simple boundary condition which is concluded from thermodynamics: the entropy of a system must be vanish at absolute zero temperature β. that is we must have:

$$\lim_{\beta\to\infty}(\beta^2\frac{\partial\omega}{\partial\beta}) = 0$$

Thus we must limit ourselves only to $a < -1$. But the coefficients remain undetermined. Thus we take another endeavor to solve this equation. We substitute (33) in (37) and take derivatives and equalize from both sides all terms which have the same order of m^2 (i.e.; the coefficients of $(-\frac{\partial}{\partial m^2})^j$, $j = 0, 1, 2$ We have:

$$\int_0^\infty \frac{\epsilon g(\epsilon)}{e^{\beta(\mu-\epsilon)} - 1}d\epsilon = -\frac{1}{\beta}\int_0^\infty \ln(1 - ze^{-\beta\epsilon})g(\epsilon)d\epsilon \tag{31}$$

We assume that $g(\epsilon) = \frac{d\sigma(\epsilon)}{d\epsilon}$ and also we suppose that $\sigma(\epsilon)$ is a well behavior function of energy ϵ at the ground state energy level $\epsilon = 0$. After integrating part by part and expanding both sides in terms of z $(0 < z < 1)$ and set equal all terms of same z power we have:

$$\int_0^\infty e^{-\beta n\epsilon}[\sigma(\epsilon) - \epsilon\frac{d\sigma}{d\epsilon}]d\epsilon = \frac{\sigma(0)}{\beta n}, n \geq 1 \tag{32}$$

We define Laplace transform of function $\sigma(\epsilon)$ as a function of parameter $s = \beta n$ by:

$$\Sigma(s) = \int_0^\infty e^{-s\epsilon}\sigma(\epsilon)d\epsilon \tag{33}$$

We arrive at the next differential equation for transformed function:

$$\Sigma(s) + \frac{\partial}{\partial s}(s\Sigma(s) - \sigma(0)) = \frac{\sigma(0)}{s} \tag{34}$$

Which has solution:

$$\Sigma(s) = \frac{\sigma(0)}{s} + \frac{C_1}{s^2} \tag{35}$$

In this function C_1 is an unknown constant that can be determined using initial conditions. But we take it as a constant and find it's value some later. Because we solve the

partial differential equation (29), then the result can be a function of other quantities like mass of particles. The mass is not an independent thermodynamic variable. If we assume the mass as a thermodynamic variable, then the derivative of mass with respected to the density, temperature or chemical potential must be occurred in fundamental thermodynamic relations [19].

By taking inverse Laplace transformation from (35) we obtain:

$$\sigma(\epsilon) = \sigma(0) + C_1\epsilon \tag{36}$$

We mention here that for a Boson's system the chemical potential is always negative namely $-\infty < \mu \leq 0$. Since $g(\epsilon) = \frac{d\sigma(\epsilon)}{d\epsilon}$ we finally obtain:

$$g(\epsilon) = C_1 \tag{37}$$

This relation is some eccentric. Substituting (37) in definition of $g(\epsilon)$ [2] we lead to the following dispersion relation for DE:

$$k^3 = 6\pi^2 C_1\epsilon - 6\pi^2 C_2 \tag{38}$$

We know that this is not a dispersion relation for a non interactive matter field.

But there is no simple system which has such distribution. If we assume that the energy levels of such Bosonic system must be positive and the momentum of any particle also be positive, then the energy levels of this system must satisfy the following inequality:

$$\epsilon \geq \frac{C_2}{C_1} \tag{39}$$

Without any dismay from loss of generality we can take $C_2 = 0$ .With this chooses we accept that our ground state energy level is located at $\epsilon = 0$. Another pickup is tagging this energy level a non zero value. This situation could be solved by a shift of any energy value up to this value.

Substituting (37) in (25) and making use from:

$$Poly\log(2, z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2},$$

finally we obtain next function for the density of grand thermodynamic potential:

$$\omega(\mu, R, \beta) = \frac{1}{\beta^2} Polylog(2, z) \sum_{j=0}^{2} a_j(R)(-\frac{\partial}{\partial m^2})^j C_1 \tag{40}$$

Now we calculate the density of the average of the energy .We can label it as ρ_Λ which is:

$$\rho_\Lambda = \frac{\bar{E}}{V} = \int_0^{\infty} \frac{g(\epsilon)\epsilon}{e^{\beta(\mu-\epsilon)} - 1} d\epsilon$$

---

[2] Note that V is disappeared since we calculate density.

After substituting (37) in it and doing the integration we obtain:

$$\rho_\Lambda = \frac{C_1}{\beta^2} Polylog(2, z) \tag{41}$$

The mean value of the number of the particles in the unit volume of system is:

$$n = \frac{\bar{N}}{V} = \int_0^\infty \frac{g(\epsilon)}{e^{\beta(\mu-\epsilon)} - 1} d\epsilon = -\frac{C_1}{\beta} \ln(1 - z) \tag{42}$$

Since $0 < z < 1$ the value of n is not negative. Using $s = \frac{S}{V} = -\frac{\partial \omega}{\partial T}$ we have:

$$s = -\frac{2}{\beta} Poly \log(2, z) \sum_{j=0}^2 a_j(R) (-\frac{\partial}{\partial m^2})^j C_1 \tag{43}$$

Because the mass dependence of $g(\epsilon)$ is undetermined , so we can not conclude strongly about the sign of entropy. It can be negative or positive [14].

Negative entropy is problematic if we accept that the entropy is in the association with the measure of the number of microstates in statistical mechanics. The intuition of statistical mechanics requires that the entropy of all physical components to be positive. Besides if we consider the universe as a thermodynamic system, the total entropy of the universe including DE and DM should satisfy the second law of thermodynamics. The GSL for phantom and non-phantom DE has been explored in [12]. It was found that the GSL can be protected in the universe with DE. The GSL of the universe with DE has been investigated in [13,25,26,27] as well. [3] It was found that the equation of state of the DE is uniquely determined and the phantom entropy is negative [13]. For the phantom, we either run into negative entropy problem or the GSL is violated.[4] Considering the apparent horizon as the physical boundary of the universe, it was found that both the temperature and entropy can be positive for the DE, including phantom. But in the general framework that we discussed, where the cosmological metric has a slow varying time dependency, we can assume that the time contiguity of volume and temperature is venial and for this reason we obtained a negative entropy. We must note here that if in an ensemble of particles the volume and temperature were non static, the statistical behavior of this system is not in the context of the equilibrium thermodynamics.[5]

VI: CONCLUSIONS

We have derived a grand canonical partition function for a DE's like EOS. The thermodynamic equations are generally expressed in terms of three variables: the temperature T ,

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[3] We may think the DE temperature is equal or proportional to the horizon temperature T_H.

[4] It is more realistic to consider that the physical volume and the temperature of the universe are related, since in the general situation they both depend on the scale factor a(t).It was found that the apparent horizon is a good boundary for keeping thermodynamic laws [13].

[5] As in a FRW standard model of cosmology, since our background metric is not static and thereupon the apparent horizon is time variable, then construction of a local quantum statistical and counting all microstates must affect the negative entropy and we can deduce a statistical proof for non negativity and non violation of GSL like Gong and collaborations proof [6].

chemical potential μ associating with a conserved particle number N, and scalar curvature R. We also reviewed some known results for an ideal Bose gas in curved spacetime. Comparing our result for a negative pressure matter with an ideal gas we concluded that there is a significance generic difference between the statistical distribution of negative pressure matters and ideal gases. Specifically , the dispersion relation for a dark energy like EOS is non quadratic (in contradiction with a non interactive ideal Bose gas)and comparing this dispersion function with the statistical model of SQM we observe that also in SQM systems this dispersion relation is valid in some physically accessible region of the system[19].

Also we have derived the mean value of the energy of the system in terms of the curvature . We show that if a matter field (Boson) obeys an EOS like Dark energy and in cosmological language violates the well known energy conditions in General relativity , the degeneracy function of this gas must be an implicit constant function of energy. This function implicitly is dependent on the mass of the particles which is not a thermodynamic quantity and remains as an undetermined function in this formula.

The derived thermodynamic equations has show that the entropy may be positive or negative. As it was stated in [6], this negativity of the entropy is hidden behind this significance assumption that volume and temperature are independent. But we know that in an expanding FRW universe both are functions of time. Hence if we assume that our metric is quasi static or has a negligible dynamics, any phantom field of DE with an intelligible statistical testimony has a negative entropy. If we want to generalize our argument to a realistic cosmological case, we must work in a non equilibrium statistical regime.

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